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DESIGN CRITERION FOR CIRCUMFERENTIAL RING STIFFENERS FOR A CONE LOADED BY EXTERNAL PRESSURE

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TITLE

DESIGN CRITERION FOR CIRCUMFERENTIAL RING STIFFENERS
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OBJECT

To derive basic ring load solutions acting on a conical shell and to apply these results to a des[illegible] criterion for circumferential ring stiffeners for a cone loaded by external pressure.

ABSTRACT

1. Circumferential line loading (i.e., "ring" loading)solutions for normal and shear force distributions acting externally on a cone are carried out in detail in the Appendix. By superposition, the corresponding solution for a "ring" load of horizontal force was obtained.

2. By making suitable assumptions, the results in paragraph 1 were applied to the formulation of a design criterion for circumferential ring stiffeners on a cone loaded by external pressure. This procedure is developed in Section II of the report.

3. A summary of the pertinent design information including the reaction force on a ring stiffener, spacing of the rings, and stresses in the cone under a ring is presented in Section III.

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I. INTRODUCTION

Although considerable attention has been devoted in the literature to the study of cylindrical shells, the somewhat more difficult problem of a conical shell has received far less attention. In particular, information necessary for the design of circumferential ring stiffeners for conical shells does not appear readily available.

This report contains a derivation of the stress analysis corresponding to the two cases of ring loads, external pressure and external shear. A design criterion for circumferential rings with rectangular(1) cross-section is carried out which utilizes the results of the basic stress analysis.

It is believed that the design criterion provided in this report represents a considerable improvement over that currently being used. However, the solution must be considered as only an approximation to the ring stiffeners contemplated due to the present simplification of the cross-sectional geometry of the stiffeners. Should a more precise analysis of a ring stiffener with a more complicated cross-sectional geometry be desired, much of the information already obtained in this report can be utilized.

Development of a design criterion for circumferential ring stiffeners for cones is carried out in the first part of this report. The detailed stress analysis for the ring loading of a cone is carried out in the Appendix.

(1) Actually, only the cross-sectional area is utilized. If only the primary effects are considered, i.e., the average hoop stress in the ring, the criterion is applicable to cross-sections with any geometry.

II. DEVELOPMENT OF A DESIGN CRITERION FOR CIRCUMFERENTIAL RING STIFFENERS FOR A CONE UNDER EXTERNAL PRESSURE.

A. Statement of the Problem.

Consider a circular cone whose cross-section is illustrated in Figure 1.

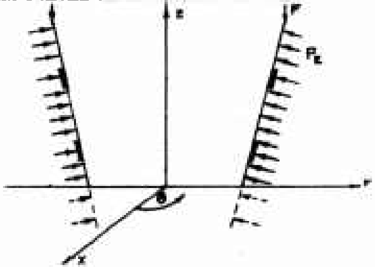

FIGURE 1

The cone is acted upon by external pressure P_E, and the net vertical force is considered to be balanced by the force F applied to the upper end, as shown in Figure 1. Circumferential ring stiffeners with cross-sections illustrated by the shaded bands are to be spaced along the slant length[2] of the cone.

Although the lower section of the cone can be closed, we do assume that the rings are spaced at sufficient distances from the vertex and the base support of the cone, and from each other so that end effects or interaction effects are negligible.

(2) See loc. cit.

B. Radial Deflection of a Conical Shell under External Pressure, P_E.

Consider the deformation of a conical shell (with no stiffeners) under external pressure, P_E, as illustrated in Figure 2.

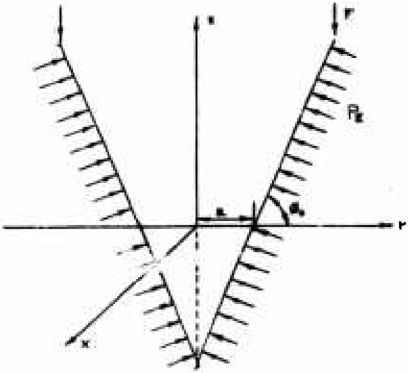

FIGURE 2

Let us consider the deformation at $r = a$, and define

$$\tan \phi_o = 1/P_1 . \qquad \ldots\ldots (1)$$

The horizontal or radial displacement, u, (positive in the increasing r -direction can be found in the Appendix. In particular, from (A-110),

$$u_{r=a} = \frac{-a^2 P_E \sqrt{1 + P_1^2}}{Eh} (1-\nu/2). \qquad \ldots\ldots (2)$$

C. A Simplified Design Criterion.

A simplified criterion for the design of reinforcing circumferential rings for a cone under external pressure is the assumption that the reactive forces between the ring and the cone correspond to a uniform radial tension, F_r. Such an assumption neglects, of course, the effect of reaction forces which cause bending of the ring. In fact, such an assumption permits only enough flexibility to ensure compatible radial displacements.

Subject to the limitations of such a simplified assumption of the reaction forces, we now proceed to formulate a design criterion on this basis. Let δ_1 be the decrease of the radius $r = a$ due to the external pressure, P_E, acting on the non-reinforced cone. Then, from (2),

$$\delta_1 = + \frac{a^2 P_E \sqrt{1 + p_1^2}\ (1 - \nu/2)}{E_s h} . \qquad \ldots . (3)$$

(Note that in the limiting case of a cylinder, $p_1 = 0$, we consider the pressure acting on the ends as well as the external curved surface.)

Now, let the magnitude of the radial reactive force per unit length of the circumference of the cone at $r = a$ be denoted by F_r.

The reactive force acting on the ring (assuming the cross-sectional dimensions of the ring are small in comparison with the radius, a, produces an average compressive force of $F_r a$. Thus, the corresponding decrease of the inner radius of the ring is approximately:

$$\delta_2 = \frac{F_r a^2}{A E_r} , \qquad \ldots . (4)$$

where A is the cross-sectional area of the ring.

Finally, the reactive force acting on the cone causes an extension of the radius by an amount δ_3. In particular, from Equation (A-97),

$$\delta_3 = \left\{ \frac{a\lambda \sqrt[4]{1 + p_1^2}}{2} - \frac{(1 + 4\nu)^2 p_1^2}{64 a\lambda \sqrt[4]{1 + p_1^2}} \right\} \frac{a F_r}{E_s h \sqrt{1 + p_1^2}} \qquad \ldots . (5)$$

For compatibility of displacements,

$$\delta_1 = \delta_2 = \delta_3 \, . \qquad \ldots\ldots (6)$$

Since $a\lambda = \sqrt{\frac{a}{h}} \sqrt[4]{3(1-\nu^2)}$, it is evident that $1/a\lambda$ is negligible compared with $a\lambda$ and the second term of the bracketed expression in (5) is negligible for the range of cone angles in which we are interested. Therefore,

$$\frac{F_r\, a^2}{A\, E_r} + \frac{F_r\, a^2 \lambda}{2E_s h \sqrt[4]{1 + p_1^2}} = \frac{a^2\, P_E \sqrt{1 + p_1^2}\,(1-\nu/2)}{E_s h} \, ,$$

or

$$F_r \left\{ \frac{\lambda}{2hE_s \sqrt[4]{1 + p_1^2}} + \frac{1}{A\, E_r} \right\} = P_E \frac{\sqrt{1 + p_1^2}\,(1-\nu/2)}{E_s h} \, . \qquad \ldots\ldots (7)$$

For the case of the same material for the cone and stiffener, $E_s = E_R = E$, thus,

$$F_r \left\{ \frac{\lambda}{2h \sqrt[4]{1 + p_1^2}} + \frac{1}{A} \right\} = \frac{P_E \sqrt{1 + p_1^2}\,(1 - \nu/2)}{h} \, . \qquad \ldots\ldots (8)$$

(It is interesting to note that this formula coincides with the solution for a cylinder, $p_1 = 0$, as found in Timoshenko, "Theory of Plates and Shells", pp. 405-406.)

D. Spacing of Stiffeners Necessary to Avoid Interaction Considerations.

In order to estimate the effect of the proximity of adjacent stiffeners in influencing the design criterion (7), we consider the rapidity in which the radial point load dies out with distance away from the point of application. For the range of cones which we are interested in, it can be shown, after some algebra and order of magnitude consideration, that

$$\frac{u(\text{at } r)}{u(\text{at } a)} \approx \sqrt[4]{\frac{r}{a}}\; e^{-G} (\cos G + \sin G) \, , \qquad \ldots\ldots (9)$$

where G is defined in the Appendix.

By examining the numerical values of this ratio, we can draw the following conclusions:

A conservative estimate of the spacing of circumferential ring stiffeners such that the design criterion (7) can be applied independent from appreciable interaction effects can be simply stated. Consider a stiffener with radius "a" and let "d" be the slant distance between the given stiffener and its adjacent neighbor. If interaction of 5 percent of the radial displacement is tolerated, then:

Case 1, if the stiffener has only one adjoining neighbor, we should have

$$d \geq \frac{2}{\lambda\sqrt{\sin \phi_o}} \quad , \qquad \dots . (10)$$

and

Case 2, if the stiffener has adjoining neighbors on both sides; i.e., it is a central stiffener of three equally spaced stiffeners,

$$d \geq \frac{4}{\lambda\sqrt{\sin \phi_o}} \quad . \qquad \dots . (11)$$

III. SUMMARY OF A DESIGN CRITERION FOR CIRCUMFERENTIAL RING STIFFENERS ON A CONE LOADED BY EXTERNAL PRESSURE

Subject to the approximations discussed in the preceding sections, the corresponding design criterion for circumferential ring stiffeners on a cone loaded by external pressure can be summarized as follows:

A. Reactive Force on the Ring.

The reactive load acting between the ring stiffener and the cone is determined in terms of the applied pressure from

$$F_r\left\{\frac{\lambda\sqrt{\sin \phi_o}}{2h\, E_s} + \frac{1}{A E_r}\right\} = \frac{P_E\,(1-\sigma^2)}{h E_s \sin \phi_o} , \qquad \dots . (12)$$

where

P_r = Reactive force per unit length of circumference (acting horizontally).

P_z = External pressure acting on the cone

h = Thickness of the conical shell.

ν = Poisson's ratio for the cone.

E_S = Young's Modulus for the cone.

E_γ = Young's Modulus for the ring.

A = Cross-sectional area of the ring.

ϕ_o = Cone angle (Figure 2)

$\lambda = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{ah}}$ where a is the radius of the cone at the location of the ring stiffener.

B. Spacing of the Ring Stiffeners.

An error of less than 5 percent is made in using (12) for multiple stiffeners, provided the stiffeners are located "d" distance apart where d is measured along the slant length. The value of d is given by

$$\text{Case 1,} \qquad d \geq \frac{2}{\lambda\sqrt{\sin\phi_o}} \qquad \ldots\ldots (13)$$

if the stiffener has only one adjoining neighbor.

$$\text{Case 2,} \qquad d \geq \frac{4}{\lambda\sqrt{\sin\phi_o}} \qquad \ldots\ldots (14)$$

if the stiffener has adjoining neighbors on both sides; i.e., it is a central stiffener of three equally spaced stiffeners.

C. Stresses in the Cone under the Ring Stiffener.

The stresses in the cone under the ring stiffener are obtained by superposition of the effects due to the external pressure, P_E, and the reaction force, F_r. This information is contained in the Appendix, Equations (A-90) through (A-109). Considering the range of parameters which is of practical interest, one can simplify these expressions from the order of magnitude arguments. The following approximate formulas for the stresses in the cone under a stiffener are obtained:

1. Bending moment resultant, M_ξ .

$$M_\xi \approx \frac{F_r \sqrt{\sin \phi_o}}{4\lambda} \qquad \ldots . (15)$$

2. Shear stress resultant, Q .

$$Q \approx \mp 1/2\, F_r \sin \phi_o . \qquad \ldots . (16)$$

3. Stress resultant, N_ξ .

$$N_\xi \approx \mp 1/2\, F_r \cos \phi_o - \frac{P_E\, a}{2 \sin \phi_o} \qquad \ldots . (17)$$

4. Stress resultant, N_θ .

$$N_\theta \approx \left\{ \pm \frac{\nu \cos \phi_o}{2} - \frac{a\lambda \sqrt{\sin \phi_o}}{2} \right\} F_r - \frac{aP_E}{\sin \phi_o} . \qquad \ldots . (18)$$

5. Meridional bending stress, $\sigma_{\xi B}$

$$\sigma_{\xi B} \approx - \frac{3\sqrt{\sin \phi_o}}{2\lambda h^2} F_r . \qquad \ldots . (19)$$

Definitions of the stress resultants, etc. are given in the first section of the Appendix.

ACKNOWLEDGMENT

The derivations and presentation of formulas have been reviewed by Ralph F. Julian, member of the Mathematics Section, Watertown Arsenal Laboratories, Watertown Arsenal.

Appendix

We now summarize the analysis leading to the solution for ring loading of a right circular cone. Such information is intended to provide the designer the underlying assumptions and approximations used in obtaining the solution and a review of the nomenclature used.

I. Basic Equations of a Linearized Thin-Shell Theory for Axially Symmetric Deformation.

In this section, we shall summarize the basic equations of a linearized thin-shell theory as formulated by E. Reissner [1]. Due to the axial symmetry of the geometry and the applied load the basic equations will be directly modified to the case of axially symmetric deformation.

1. Definition of Stress Resultants and Couples.

It is convenien. to introduce curvilinear coordinates for the description of the middle surface of the shell. A point P on the shell of revolution can be described by two parameters ξ and θ as shown in Figure A-1. A point P' lying in the shell but not on the middle surface can be described relative to the middle surface by a third parameter ζ measured along the normal to the middle surface passing through P'. Under obvious physical restrictions of the shell, the parameters ξ, θ, ζ constitute an orthogonal curvilinear coordinate system which describes uniquely every point in the shell.

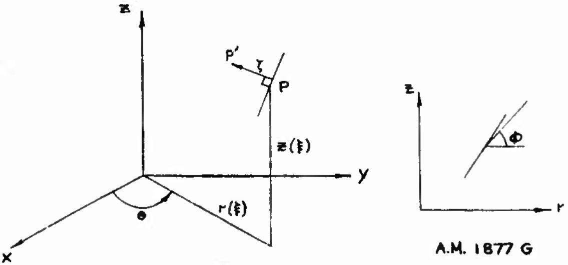

FIGURE A-1

Now consider a curvilinear element of the shell as shown in Figure A-2. The thickness of the shell, h, will be assumed much smaller than the principal radii of curvature of the curvilinear element of the middle surface; hence, the variation of the lengths of the sides of the element in the thickness direction will be neglected. The coefficient α is found from the relation

$$\alpha^2 = (r')^2 + (z')^2, \qquad \text{(A-1).}$$

where primes denote differentiation with respect to ξ.

For axially symmetric deformation the only nontrivial stress components are σ_ξ, σ_θ, σ_ζ, $\tau_{\xi\zeta}$. The resultant stresses and couples due to these stresses when h is small compared with the principal radii of curvature are defined as follows:

$$
\begin{aligned}
N_\xi &= \int_{-h/2}^{h/2} \sigma_\xi \, d\zeta \\
N_\theta &= \int_{-h/2}^{h/2} \sigma_\theta \, d\zeta \\
Q &= \int_{-h/2}^{h/2} \tau_{\xi\zeta} \, d\zeta \\
M_\xi &= \int_{-h/2}^{h/2} \zeta \sigma_\xi \, d\zeta \\
M_\theta &= \int_{-h/2}^{h/2} \zeta \sigma_\theta \, d\zeta
\end{aligned}
\qquad \text{(A-2)}
$$

It is convenient to introduce the additional notation defined in Figure A-3. The external stress vector considered as acting on the middle surface (due to the thinness of the shell) will be considered as resolved into a vertical component, P_V, and a horizontal component, P_H. Furthermore, horizontal and vertical stress vectors H and V, respectively, will be introduced whereby

$$
\begin{aligned}
N_\xi &= H \cos\phi + V \sin\phi \\
Q &= -H \sin\phi + V \cos\phi
\end{aligned}
\qquad \text{(A-3).}
$$

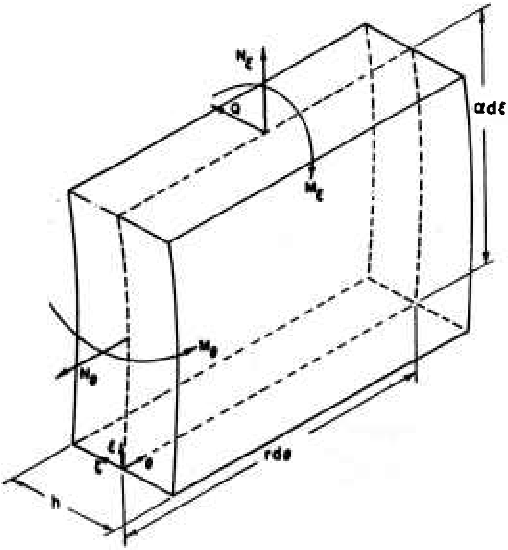

FIGURE A-2

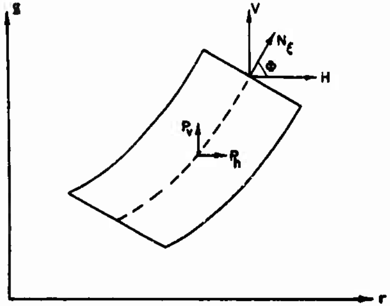

FIGURE A-3

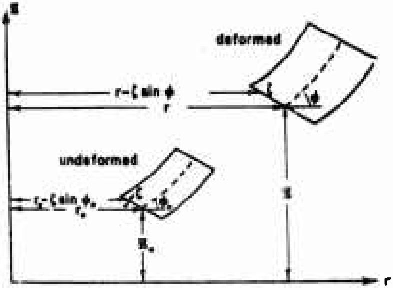

FIGURE A-4

2. Strain Components

To describe the components of strain it is convenient to refer to Figure A-4. Subscripts "o" are used to describe the element in its undeformed position. Furthermore, in addition to the smallness of h, a second basic assumption is made - namely, a normal element to the undeformed middle surface is carried over (without change in length) into a normal to the deformed middle surface.

The displacements in the radial and axial directions are denoted by u and w, respectively. Thus

$$r = r_o + u \qquad \text{(A-4)}$$

$$z = z_o + w$$

The two principal strains, ϵ_ξ and ϵ_θ, can be defined in the usual manner of thin shell theory in which at most a linear variation of the strain across the thickness is assumed. Thus

$$\epsilon_\theta = \epsilon_{\theta m} + \zeta K_\theta \qquad \text{(A-5)}$$

$$\epsilon_\xi = \epsilon_{\xi m} + \zeta K_\xi$$

3. Linearized Theory - Basic Relations

The linearized theory which will now be considered originates by setting

$$\phi = \phi_o - \beta \qquad \text{(A-6)}$$

and linearizing with respect to β. Furthermore, in the equilibrium equations no distinction is made between the deformed and the undeformed element. The basic relations which are used in the linearized theory are as follows:

Horizontal and Vertical Stress Resultants

$$N_\xi = H \cos\phi_o + V \sin\phi_o \qquad \text{(A-7)}$$

$$Q = -H \sin\phi_o + V \cos\phi_o$$

Force and Moment Equilibrium Equations

$$(rV)' + r\alpha P_V = 0$$

$$(r\,H)' - \alpha N_\theta + r\alpha P_H = 0 \tag{A-8}$$

$$(r\,M_\xi)' - \alpha(\cos\phi_o)\,M_\theta + r\alpha(H\sin\phi_o - V\cos\phi_o) = 0$$

Linearized Strain Components

$$\epsilon_{\theta m} = u/r_o \tag{A-9}$$

$$\epsilon_{\xi m} = \frac{u'}{\alpha_o} - \beta(\tan\phi_o)\left(1 + \frac{w'}{\alpha_o}\right)$$

and

$$K_\xi = \beta'/\alpha_o \tag{A-10}$$

$$K_\theta = (\beta\cos\phi_o)/r_o$$

$$w' = z_o'\,\epsilon_{\xi m} - r_o'\beta \tag{A-11}$$

Stress - Strain Relations

$$N_\xi = \frac{Eh}{1-\nu^2}\left(\epsilon_{\xi m} + \nu\,\epsilon_{\theta m}\right)$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_{\theta m} + \nu\,\epsilon_{\xi m}\right) \tag{A-12}$$

$$M_\xi = D\,(K_\xi + \nu\,K_\theta)$$

$$M_\theta = D\,(K_\theta + \nu\,K_\xi)$$

$$D = Eh^3/12(1-\nu^2)$$

Where ν and E are Poisson's ratio and Young's modulus, respectively.

<u>Compatibility Equation</u>

$$r_0' \epsilon_{\xi m} - (r_0 \epsilon_{\theta m})' = - z_0' \beta \qquad \text{(A-13)}$$

4. Reduction to Two Simultaneous Equations

The basic relations summarized in the previous section can be combined into two simultaneous differential equations. For the case of constant shell thickness, h, this system can be written as,

$$\beta'' + \frac{(r/\alpha)'}{(r/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \frac{\nu(r'/\alpha)'}{(r/\alpha)}\right]\beta + \frac{\alpha m z'}{rh}\psi = \frac{\alpha r'}{rD}(rV)$$

$$\psi'' + \frac{(r/\alpha)'}{(r/\alpha)}\psi' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]\psi \qquad \text{(A-14)}$$

$$- \frac{\alpha m z'}{rh}\beta = -\left\{\left[\frac{(r/\alpha)'}{(r/\alpha)} + \nu\frac{r'}{r}\right]\frac{\alpha r m}{Eh^2}P_H + \frac{m}{Eh^2}(r\alpha P_H)'\right\}$$

$$+\left\{\left[\frac{r' z'}{r^2} + \frac{\nu(z'/\alpha)'}{r/\alpha}\right]\frac{m}{Eh^2}(rV) + \frac{\nu m z'}{Eh^2 r}(rV)'\right\}$$

where

$$m^2 = 12(1-\nu^2)$$

and

$$\psi = \frac{m r H}{E h^2}.$$

II. General Solution of the Linearized Theory for a Cone.

1. Curvilinear Coordinates for the Cone.

The particular curvilinear coordinate system for the case of a conical shell is indicated in Figure A-5.

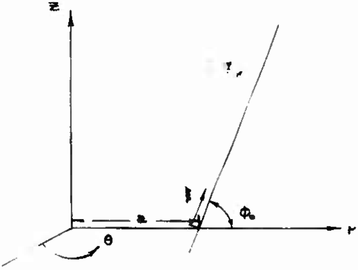

FIGURE A-5

The middle surface of the cone is described by θ and

$$r = p_1 a \xi + a$$
$$z = a \xi \qquad \text{(A-15)}$$

where

$$\tan \phi_0 = 1/p_1 \qquad \text{(A-16)}$$

It is evident that $r'/z' = p_1$ and $\alpha = a\sqrt{1+p_1^2}$.
Furthermore, $\cos\phi_o = p_1/\sqrt{1+p_1^2}$ and $\sin\phi_o = 1/\sqrt{1+p_1^2}$

2. General solution of the basic linearised equations.

In order to solve the basic linearised equations it is necessary to find a solution to the system of equations, (A-14). This is done by combining a particular solution of the nonhomogeneous system with the general solution of the homogeneous system.

The "membrane" solution is taken as the particular solution of the nonhomogeneous system. In particular, denoting by the subscript "m", the quantities corresponding to the membrane approximation are

$$\beta_m = 0$$
$$\psi_m = \frac{m}{Eh^2}\left(\frac{r'}{z'}\right)(rV) = \frac{m p_1}{Eh^2}(rV)$$

The solution of the homogeneous system of equations corresponding to (A-14) can be obtained by the well-known process of asymptotic integration. Such a solution, valid provided the cone is not too shallow, can be written as follows:

$$\left.\begin{aligned}
\beta_H &= \frac{1}{\sqrt[4]{r}}\left\{e^{-G}(A_{oc}\cos G + A_{1c}\sin G) + e^{+G}(B_{oc}\cos G - B_{1c}\sin G)\right\} \\
\psi_H &= \frac{1}{\sqrt[4]{r}}\left\{e^{-G}(A_{1c}\cos G - A_{oc}\sin G) + e^{+G}(B_{oc}\sin G + B_{1c}\cos G)\right\} \\
&\text{where} \\
G &= \frac{1}{p_1}\sqrt{\frac{\text{[illegible]}}{h}} \times \sqrt[4]{1+p_1^{2}} \times (\sqrt{r} - \sqrt{a})
\end{aligned}\right\} \quad \text{(A-18)}$$

The general solution for the conical section can be written as

$$\left.\begin{aligned}
\beta &= \beta_H + \beta_m \\
\psi &= \psi_H + \psi_m
\end{aligned}\right\} \quad \text{(A-19)}$$

Thus,

$$\beta = \frac{1}{\sqrt{r}} \left\{ e^{-\sigma} (A_{oc} \cos \sigma + A_{1c} \sin \sigma) + e^{+\sigma} (B_{oc} \cos \sigma - B_{1c} \sin \sigma) \right\}$$

$$\psi = \frac{1}{\sqrt{r}} \left\{ e^{-\sigma} (A_{1c} \cos \sigma - A_{oc} \sin \sigma) + e^{+\sigma} (B_{oc} \sin \sigma + B_{1c} \cos \sigma) \right\} + \frac{mr_1}{Eh^2}(rV) \qquad \text{(A-20).}$$

The constants in (A-20) must be determined from the end or junction conditions of the particular problem considered.

III. Solution for Ring Load of Normal Pressure.

For "ring" loading of external normal pressure we choose our coordinate description as is indicated in Figure A-6.

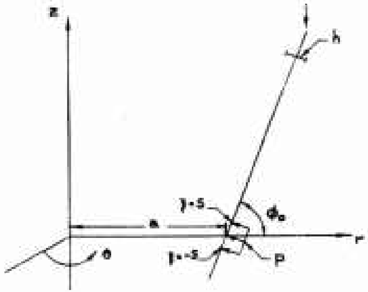

FIGURE A-6.

The vertical and horizontal components in the convention defined in Figure A-3 are therefore

$$P_V = P\cos\phi_0 \qquad -\delta \le \xi \le +\delta \qquad \text{(A-21)}$$

$$= 0 \qquad \delta < |\xi| < \infty$$

$$P_H = -P\sin\phi_0 \qquad -\delta \le \xi \le +\delta \qquad \text{(A-22)}$$

$$= 0 \qquad \delta < |\xi| < \infty$$

Since the applied load is not self-equilibrating, it is necessary to provide an additional support at the ends. Since we are only concerned with the local effects of the ring load, without loss of accuracy we shall assume that the unbalanced applied load is balanced by a vertical load acting at $\xi = \infty$.

Hence, we assume

$$(rV) = 0 \qquad \xi \leq -s. \tag{A-23}$$

From the first of the equilibrium equations, (A-8), we find

$$(rV) = -\int^{\xi} r\alpha P_v \, d\xi = -\int^{\xi} rr' P d\xi$$

$$= 0 \ , \qquad \xi \leq -s \tag{A-24}$$

$$= \frac{-P}{2} \left\{ \left[r(\xi) \right]^2 - \left[r(-s) \right]^2 \right\}, \quad -s \leq \xi \leq s$$

$$= -2p_1 \, sPa^2 \ , \ \xi > s$$

Since the solution must be bounded at top and bottom of the cone it can be seen from (A-20) that (subscript "n" for normal "point load")

$$\beta = \frac{1}{\sqrt{r}} \left\{ e^{-G} \left(A_{on} \cos G + A_{in} \sin G \right) \right\}$$

$$\psi = \frac{1}{\sqrt{r}} \left\{ e^{-G} \left(A_{in} \cos G - A_{on} \sin G \right) \right\} + \frac{mp_1}{Eh^2} (rV) \tag{A-25}$$

for $\xi > 0$,

and,

$$\beta = \frac{1}{\sqrt{r}} \left\{ e^{+G} \left(B_{on} \cos G - B_{in} \sin G \right) \right\} \tag{A-26}$$

$$\psi = \frac{1}{\sqrt{r}} \left\{ e^{+G} \left(B_{on} \sin G + B_{in} \cos G \right) \right\} + \frac{mp_1}{Eh^2} (rV)$$

for $\xi < 0$.

The four constants A_{on}, A_{in}, B_{on} and B_{in} must be determined from conditions at $\xi = 0$. From obvious physical considerations we require continuity of β and β' at $\xi = 0$. These conditions impose the restrictions, respectively,

$$A_{on} = B_{on} \tag{A-27}$$

and

$$A_{in} + B_{in} = 2A_{on} \ . \tag{A-28}$$

The two remaining conditions must be obtained by force considerations at $\xi = 0$.

Since on physical grounds it is evident that N_ξ must be continuous at $\xi = 0$, it follows from (A-8) and (A-12) that the quantity N_ϕ must also be continuous at $\xi = 0$. Continuity of N_ξ implies continuity of (rN_ξ) which in turn from (A-7) implies continuity of $(rH)\cos\phi_0 + (rV)\sin\phi_0$. Therefore,

$$\lim_{s\to 0}\left\{\left[(rH)_{\xi=s} - (rH)_{\xi=-s}\right]\cos\phi_0\right\} = \lim_{s\to 0} - \left\{(rV)_{\xi=s} - (rV)_{\xi=-s}\right\}\sin\phi_0 \tag{A-29}$$

From (A-14), (A-16), and (A-24), it follows that

$$\lim_{s\to 0} \frac{Eh^2 p_1}{m}\left[\psi_{\xi=s} - \psi_{\xi=-s}\right] = \lim_{s\to 0} (2p_1 s P a^2). \tag{A-30}$$

From (A-25) and (A-26),

$$\lim_{s\to 0}\left[\psi_{\xi=s} - \psi_{\xi=-s}\right] = \frac{1}{\sqrt[4]{a}}\left(A_{1n} - B_{1n}\right) + \frac{m p_1}{Eh^2}\lim_{s\to 0}\left(-2p_1 s P a^2\right) \tag{A-31}$$

Thus,

$$\frac{Eh^2}{m\sqrt[4]{a}}\left(A_{1n} - B_{1n}\right) = \lim_{s\to 0} 2a^2 s P\left(1 + p_1^2\right) \tag{A-32}$$

The remaining condition is found from an examination of the last two equilibrium conditions in (A-8). Again from continuity it follows that $(rH)'$ and $(rN_\xi)' + r\alpha(H\sin\phi_0 - V\cos\phi_0)$ must be continuous at $\xi = 0$. Thus, since the continuity of $(rH)'$ implies the continuity of ψ', we have the condition

$$\lim_{S\to 0} \psi'_{\xi=+S} = \lim_{S\to 0} \psi'_{\xi=-S} \qquad \text{(A-33)}$$

Applying this condition to (A-25) and (A-26) and utilising (A-28), yields

$$p_1(A_{in} - B_{in}) = -16\sqrt{\frac{am}{2h}}\ \sqrt[4]{1+p_1^2}\ A_{on}. \qquad \text{(A-34)}$$

The continuity of $(rM_\xi)' + r\alpha(H \sin\phi_0 - V \cos\phi_0)$ can be shown to follow from the preceding conditions.

The total surface area over which the applied load is acting is $4\pi aS\sqrt{1+p_1^2}$. Thus, if we define F_n as force per lineal inch of ~~the~~ circumference of the cone at $\xi = 0$,

$$F_n = 2aS\sqrt{1+p_1^2}\ P \qquad \text{(A-35)}$$

By definition of the ring load concept, we therefore write

$$\lim_{S\to 0} 2aS\sqrt{1+p_1^2}\ P = F_n \qquad \text{(A-36)}$$

We therefore have the four conditions:

$$\begin{aligned}
&A_{on} = B_{on}\\
&A_{in} + B_{in} = 2A_{on}\\
&A_{in} - B_{in} = \frac{m\sqrt[4]{a}}{Eh^2}\ a\sqrt{1+p_1^2}\ F_n\\
&p_1(A_{in} - B_{in}) = -16\sqrt{\frac{am}{2h}}\ \sqrt[4]{1+p_1^2}\ A_{on}
\end{aligned} \qquad \text{(A-37)}$$

(we note that for the cylindrical case, $p_1 = 0$; hence $A_{on} = 0$ and the well-known cylindrical result is obtained.)

The system (A-37) leads to the following values of the coefficients:

$$\left.\begin{aligned}
A_{on} = B_{on} &= \frac{-p_1\sqrt[4]{1+p_1^2}\ \sqrt[4]{a}}{16Eh}\sqrt{\frac{2am}{h}}\ F_n\\
A_{in} &= -\left\{\frac{p_1\sqrt[4]{1+p_1^2}\ \sqrt[4]{a}}{16Eh}\sqrt{\frac{2am}{h}} - \frac{am\sqrt[4]{a}\sqrt{1+p_1^2}}{2Eh^2}\right\} F_n\\
B_{in} &= -\left\{\frac{p_1\sqrt[4]{1+p_1^2}\ \sqrt[4]{a}}{16Eh^2}\sqrt{\frac{2am}{h}} + \frac{am\sqrt[4]{a}\sqrt{1+p_1^2}}{2Eh^2}\right\} F_n
\end{aligned}\right\} \qquad \text{(A-38)}$$

The solution is now complete in that all information can be obtained from the functions β and ψ which are now completely defined by Equations (A-25), (A-26), and (A-38).

IV. Solution for Ring Load of Shear.

For "ring" loading of external shear we choose our coordinate description as indicated in Figure A-7.

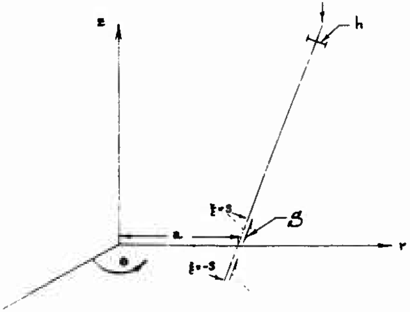

FIGURE A-7

The vertical and horizontal components of the applied load are therefore

$$P_V = \mathcal{S} \sin\phi_0 \qquad -S \le \xi \le S \qquad \text{(A-39)}$$
$$= 0 \qquad S < |\xi| < \infty$$

$$P_H = \mathcal{S} \cos\phi_0 \qquad -S \le \xi \le S \qquad \text{(A-40)}$$
$$= 0 \qquad S < |\xi| < \infty$$

Again we will assume the unbalanced vertical component of the applied load is balanced by a vertical load acting at $\xi = \infty$. From the first of the equilibrium conditions

$$
\begin{aligned}
(rV) &= -\int^{\xi} r\alpha P_v d\xi = -\int^{\xi} a\, r\, \mathcal{S}\, d\xi \\
&= 0 \quad , \qquad\qquad\qquad\qquad \xi \leq -s \\
&= -a^2 \mathcal{S}\left[\xi + s + \tfrac{p_1}{2}(\xi^2 - s^2)\right] , \quad -s \leq \xi \leq s \\
&= -2a^2 s\, \mathcal{S} \quad , \qquad\qquad\qquad \xi \geq s
\end{aligned}
\tag{A-41}
$$

From boundedness of solution, it again follows that (subscript "s" is used for shear "ring loading")

$$
\begin{aligned}
\beta &= \frac{1}{\sqrt[4]{r}}\left\{e^{-G}\left(A_{0s}\cos G + A_{1s}\sin G\right)\right\} \\
\psi &= \frac{1}{\sqrt[4]{r}}\left\{e^{-G}\left(A_{1s}\cos G - A_{0s}\sin G\right)\right\} + \frac{m p_1}{Eh^2}(rV)
\end{aligned}
\tag{A-42}
$$

for $\xi > 0$.

and

$$
\begin{aligned}
\beta &= \frac{1}{\sqrt[4]{r}}\left\{e^{+G}\left(B_{0s}\cos G - B_{1s}\sin G\right)\right\} \\
\psi &= \frac{1}{\sqrt[4]{r}}\left\{e^{+G}\left(B_{0s}\sin G + B_{1s}\cos G\right)\right\} + \frac{m p_1}{Eh^2}(rV)
\end{aligned}
\tag{A-43}
$$

for $\xi < 0$.

Imposing again the conditions of continuity of β and β' at $\xi = 0$, yields the two relations,

$$A_{0s} = B_{0s} \quad , \tag{A-44}$$

and

$$A_{1s} + B_{1s} = 2A_{0s} \ . \tag{A-45}$$

Continuity of the stress resultant Q for this case of loading is evident from physical considerations. Thus, since (rQ) must also be continuous, from (A-7) it follows that $(rH)\sin\phi_0 - (rV)\cos\phi_0$ must also be continuous, or

$$
\begin{aligned}
&\lim_{s\to 0}\left\{\left[(rH)_{\xi=s} - (rH)_{\xi=-s}\right]\sin\phi_0\right\} \\
&\quad = \lim_{s\to 0}\left\{\left[(rV)_{\xi=s} - (rV)_{\xi=-s}\right]\cos\phi_0\right\}
\end{aligned}
\tag{A-46}
$$

Utilizing (A-41), this condition can be written as

$$\frac{Eh^2}{m}\lim_{s\to 0}\left\{\psi_{\xi=s}-\psi_{\xi=-s}\right\}=-\lim_{s\to 0} 2p_1 a^2 s\,\delta. \qquad \text{(A-47)}$$

But,

$$\lim_{s\to 0}\left\{\psi_{\xi=s}-\psi_{\xi=-s}\right\}$$
$$=\frac{1}{\sqrt{a}}(A_{1s}-B_{1s})+\frac{mp_1}{Eh^2}\lim_{s\to 0}(-2a^2 s\,\delta). \qquad \text{(A-48)}$$

Substituting (A-48) into (A-47) yields the condition

$$A_{1s}=B_{1s}\ . \qquad \text{(A-49)}$$

The final condition can be obtained by considering the remaining two equilibrium equations in (A-8). Consistent with the preceding assumptions, we must ensure the continuity of $(rM_\xi)'$ and $(rH)'-\alpha N_\theta$. It can be shown that $(rM_\xi)'$ is continuous provided the previous conditions are enforced. The final condition can be stated as

$$\lim_{s\to 0}\left\{(rH)'_{\xi=s}-(rH)'_{\xi=-s}\right\}$$
$$=\lim_{s\to 0}\alpha\left\{N_{\theta(\xi=s)}-N_{\theta(\xi=-s)}\right\} \qquad \text{(A-50)}$$

Since $\epsilon_{\theta m}$ is continuous, it follows from (A-12) that

$$\lim_{s\to 0}\alpha\left\{N_{\theta(\xi=s)}-N_{\theta(\xi=-s)}\right\}$$
$$=\lim_{s\to 0}\alpha\nu\left\{N_{\xi(\xi=s)}-N_{\xi(\xi=-s)}\right\} \qquad \text{(A-51)}$$

Thus, (A-50) becomes

$$\frac{Eh^2}{m}\lim_{s\to 0}\left\{\psi'_{\xi=s}-\psi'_{\xi=-s}\right\}$$
$$=\frac{Eh^2\alpha\nu}{m}(\cos\phi_0)\lim_{s\to 0}\left\{\left(\frac{\psi}{r}\right)_{\xi=s}-\left(\frac{\psi}{r}\right)_{\xi=-s}\right\} \qquad \text{(A-52)}$$
$$+\alpha\nu(\sin\phi_0)\lim_{s\to 0}\left\{\left(\frac{rV}{r}\right)_{\xi=s}-\left(\frac{rV}{r}\right)_{\xi=-s}\right\}$$

Equations (A-44), (A-45), (A-49), and (A-52) determine that

$$A_{os} = B_{os} = A_{is} = B_{is} \tag{A-53}$$
$$= \frac{\sqrt{2h}\; a\sqrt[4]{a}\; mv\,(1+p_1^2)^{3/4}}{4Eh^2\sqrt{am}} \lim_{s\to 0} 2as\delta .$$

We again introduce the shear force per linear inch of circumference at $\xi = 0$.

$$F_s = 2as\sqrt{1+p_1^2}\,\delta \tag{A-54}$$

Then,

$$A_{os} = B_{os} = A_{is} = B_{is} \tag{A-55}$$
$$= \frac{\sqrt{2h}\; a\sqrt[4]{a}\; mv\sqrt[4]{1+p_1^2}}{4Eh^2\sqrt{am}}\, F_s$$

V. Behavior of the Conical Shell at the Ring Load Location.

In this section we shall calculate the quantities particularly useful for design information.

A. Conical Shell with Ring Load of External Normal Pressure.

The general solution of this case was carried out in Section III. Before carrying out specific calculations, some simplification is provided by introducing the well-known quantity, λ, where

$$\lambda = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{ah}} = \frac{\sqrt{m}}{\sqrt{2ah}} \qquad \text{(A-56)}$$

Then the coefficients in (A-38) can be written as

$$\begin{aligned} A_{on} = B_{on} &= \frac{-a p_1 \sqrt[4]{a(1+p_1^2)}}{8Eh} \lambda F_n \\ A_{1n} &= -\left\{ \frac{a p_1 \sqrt[4]{a(1+p_1^2)}}{8Eh} \lambda - \frac{a^2 \sqrt[4]{a} \sqrt{1+p_1^2}}{Eh} \lambda^2 \right\} F_n \\ B_{1n} &= -\left\{ \frac{a p_1 \sqrt[4]{a(1+p_1^2)}}{8Eh} \lambda + \frac{a^2 \sqrt[4]{a} \sqrt{1+p_1^2}}{Eh} \lambda^2 \right\} F_n \end{aligned} \qquad \text{(A-57)}$$

Furthermore,

$$D = \frac{Eh}{4a^2\lambda^4},$$

$$G'(0) = a\lambda \sqrt[4]{1+p_1^2}$$

1. Evaluation of bending moment $M_\xi(0)$.

$$M_\xi = D(K_\xi + \nu K_\theta) = D\left\{ \frac{\beta'}{\alpha_0} + \nu\beta \frac{\cos\phi_0}{r_0} \right\}$$

$$M_\xi(0) = \left\{ \frac{-(1-4\nu)p_1^2}{128\sqrt[4]{1+p_1^2}\, a^2\lambda^3} + \frac{\sqrt[4]{1+p_1^2}}{4\lambda} \right\} F_n \qquad \text{(A-58)}$$

When the cone is the limiting case of a cylinder then $p_1 = 0$ and $M_\xi(0)$ has the well-known value $F_n/4\lambda$.

2. Evaluation of the shear resultants, $Q(0)$.

$$\begin{aligned} rQ &= -(rH)\sin\phi_0 + (rV)\cos\phi_0 \\ &= \frac{-Eh^2}{\sqrt[4]{a}\; m\sqrt{1+p_1^2}} A_{1n}, \qquad \xi = 0^+ \\ &= \frac{-Eh^2}{\sqrt[4]{a}\; m\sqrt{1+p_1^2}} B_{1n}, \qquad \xi = 0^- \end{aligned} \qquad \text{(A-59)}$$

$$Q = \left\{ \frac{p_1}{16a\lambda\sqrt[4]{1+p_1^2}} - \frac{1}{2} \right\} F_n \quad , \quad \xi = 0^+ \qquad \text{(A-60)}$$

$$= \left\{ \frac{p_1}{16a\lambda\sqrt[4]{1+p_1^2}} + \frac{1}{2} \right\} F_n \quad , \quad \xi = 0^-$$

3. Evaluation of the stress resultant, $N_{\xi(\xi=0)}$

$$rN_\xi = rH\cos\phi_0 + rV\sin\phi_0$$

$$(rN_\xi)_{\xi=0^+} = \frac{p_1 E h^2}{\sqrt{a}\, m\sqrt{1+p_1^2}} A_m + \sqrt{1+p_1^2}\,(rV)_{\xi=0^+}$$

$$= -\left\{ \frac{p_1^2}{16\lambda\sqrt[4]{1+p_1^2}} + \frac{p_1 a}{2} \right\} F_n \quad \text{at } \xi=0 \qquad \text{(A-61)}$$

$$N_{\xi(\xi=0)} = -\left\{ \frac{p_1^2}{16a\lambda\sqrt[4]{1+p_1^2}} + \frac{p_1}{2} \right\} F_n$$

(N_ξ is continuous at $\xi = 0$)

4. Evaluation of the stress resultant, N_θ

N_θ is continuous at $\xi = 0$ and

$$N_\theta = rP_H + \frac{(rH)'}{\alpha} \qquad \text{(A-62)}$$

$$N_{\theta(\xi=0)} = \left\{ \frac{p_1^2}{64\lambda a\sqrt[4]{1+p_1^2}} - \frac{a\sqrt[4]{1+p_1^2}}{2}\lambda \right\} F_n$$

5. Evaluation of the radial displacement $u_{(\xi=0)}$

$$u_{\xi=0} = r_0(\epsilon_{\theta m})_{\xi=0} = \frac{a}{Eh}(N_\theta - \nu N_\xi)_{\xi=0} \qquad \text{(A-63)}$$

$$= \frac{a}{Eh}\left\{ \frac{(1+4\nu)p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} + \frac{\nu p_1 - a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} F_n$$

6. Evaluation of the vertical displacement $w_{(\xi=0)}$

$$w' = z_0'\epsilon_{\xi m} - r_0'\beta = \frac{a}{Eh}(N_\xi - \nu N_\theta) - a p_1 \beta$$

$$= \frac{u'}{p_1} - \beta\left(a p_1 + \frac{a}{p_1}\right)$$

$$w(\xi) - w(\infty) = \frac{u(\xi)}{p_1} + \left(a p_1 + \frac{a}{p_1}\right)\int_\xi^\infty \beta(\xi)\,d\xi + C$$

where $C = -\frac{u(\infty)}{p_1}$.

Thus, at $\xi = 0$

$$w(0) - w(\infty) = \frac{u(0) - u(\infty)}{p_1} + a\left(p_1 + \frac{1}{p_1}\right)\int_0^\infty \beta(\xi)\,d\xi. \qquad \text{(A-64)}$$

7. Maximum meridional bending stress, $\sigma_{\xi B}$

At $\xi = 0$, we assume σ_ξ varies linearly across the thickness,

$$\sigma_{\xi(\xi=0)} = -\sigma_{\xi B}\left(\frac{2\zeta}{h}\right) + \sigma_{\xi M} \qquad (A\text{-}65)$$

$$M_{\xi(\xi=0)} = -\frac{2\sigma_{\xi B}}{h}\int_{-h/2}^{h/2} \zeta^2 d\zeta = -\sigma_{\xi B}\frac{h^2}{6}$$

Thus,

$$\sigma_{\xi B} = -\left\{\frac{6(1-4\nu)p_1^2}{128\, a^2 h^2 \lambda^3 \sqrt[4]{1+p_1^2}} + \frac{3\sqrt[4]{1+p_1^2}}{2\lambda h^2}\right\} F_n \qquad (A\text{-}66)$$

8. Hoop stress, $\sigma_{\theta M}$, at $\xi = 0$

$$\sigma_{\theta M(\xi=0)} = \left(N_\theta/h\right)_{(\xi=0)}$$

$$= \left\{\frac{p_1^2}{64 a\lambda h \sqrt[4]{1+p_1^2}} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2h}\right\} F_n \qquad (A\text{-}67)$$

B. Conical Shell with Ring Load of External Shear.

The general solution of this case was carried out in Section IV. Introducing λ in (A-55),

$$A_{os} = B_{os} = A_{1s} = B_{1s}$$

$$= \frac{a\nu\sqrt[4]{a(1+p_1^2)}}{2Eh}\lambda F_s \qquad (A\text{-}68)$$

1. Evaluation of $M_\xi(0)$.

$$M_\xi = D\left\{\frac{\beta'}{\alpha_0} + \nu\beta\frac{\cos\phi_0}{r_0}\right\} = -\frac{D(1-4\nu)p_1 A_{os}}{4a\sqrt[4]{a}\sqrt{1+p_1^2}}$$

$$M_{\xi(\xi=0)} = -\frac{(1-4\nu)p_1\nu F_s}{32 a^2\lambda^3\sqrt[4]{1+p_1^2}} \qquad (A\text{-}69)$$

2. Evaluation of the shear resultants, Q(0).

For this loading the shear resultant Q is continuous at $\xi = 0$.

Thus,

$$Q_{\xi=0} = -\frac{Eh^2 A_{1s}}{a\sqrt[4]{a}\, m\sqrt{1+p_1^2}}$$

$$= -\frac{\nu F_s}{4a\lambda\sqrt[4]{1+p_1^2}} \qquad (A\text{-}70)$$

3. Evaluation of the stress resultant $N_{\xi(\xi=0)}$

$$(rN_\xi)_{\xi=0^+} = \frac{p_1 Eh^2 A_{13}}{m\sqrt[4]{a}\sqrt{1+p_1^2}} + \sqrt{1+p_1^2}\,(rV)_{\xi=0^+}$$

$$= \left\{\frac{\nu p_1}{4\lambda\sqrt[4]{1+p_1^2}} - a\right\} F_s$$

$$N_{\xi(\xi=0^+)} = \left\{\frac{\nu p_1}{4a\lambda\sqrt[4]{1+p_1^2}} - 1\right\} F_s \qquad \text{(A-71)}$$

$$(rN_\xi)_{\xi=0^-} = \frac{Eh^2 p_1 B_{13}}{m\sqrt[4]{a}\sqrt{1+p_1^2}} + \sqrt{1+p_1^2}\,(rV)_{\xi=0^-}$$

$$N_{\xi(\xi=0^-)} = \left\{\frac{\nu p_1}{4a\lambda\sqrt[4]{1+p_1^2}}\right\} F_s \qquad \text{(A-72)}$$

4. Evaluation of the stress resultant, N_θ.

$$N_\theta = r P_H + \frac{(rH)'}{\alpha}$$

$$N_{\theta(\xi=0^+)} = -\left\{\frac{p_1}{4} + 2a\lambda\sqrt[4]{1+p_1^2}\right\} \frac{\nu F_s}{4a\lambda\sqrt[4]{1+p_1^2}} \qquad \text{(A-73)}$$

$$N_{\theta(\xi=0^-)} = -\left\{\frac{p_1}{4} - 2a\lambda\sqrt[4]{1+p_1^2}\right\} \frac{\nu F_s}{4a\lambda\sqrt[4]{1+p_1^2}} \qquad \text{(A-74)}$$

5. Evaluation of the radial displacement $u_{\xi=0}$

$$u_{\xi=0} = \frac{a}{Eh}\left(N_\theta - \nu N_\xi\right)_{\xi=0}$$

$$= \left\{-\frac{\nu p_1(1+4\nu)}{16a\lambda\sqrt[4]{1+p_1^2}} + \frac{\nu}{2}\right\} \frac{aF_s}{Eh} \qquad \text{(A-75)}$$

6. Evaluation of the vertical displacement, $w_{\xi=0}$.

$$w(0) - w(\infty) = \frac{u(0)}{p_1} + a\left(p_1 + \frac{1}{p_1}\right)\int_0^\infty \beta\, d\xi - \frac{u(\infty)}{p_1} \qquad \text{(A-76)}$$

7. Maximum meridional bending stress.

$$\sigma_{\xi M(\xi=0)} = -\frac{6}{h^2} M_{\xi(\xi=0)}$$

$$= \frac{8(1-4\nu)p_1 F_s}{16a^2h^2\lambda^3\sqrt[4]{1+p_1^2}} \qquad \text{(A-77)}$$

8. Hoop stress, $\sigma_{\theta M}$ at $\xi = 0$

$$\sigma_{\theta M(\xi=0^+)} = \frac{1}{h} N_{\theta(\xi=0)}$$

$$= -\left\{\frac{p_1}{4} + 2a\lambda\sqrt[4]{1+p_1^2}\right\} \frac{\nu F_s}{4ah\lambda\sqrt[4]{1+p_1^2}} \qquad \text{(A-78)}$$

$$\sigma_{\theta M(\xi=0^-)} = -\left\{\frac{p_1}{4} - 2a\lambda\sqrt[4]{1+p_1^2}\right\} \frac{\nu F_s}{4ah\lambda\sqrt[4]{1+p_1^2}} \qquad \text{(A-79)}$$

VI. Asymptotic Approximation of the Integrals Involved in the Evaluation of $w(0)$.

It was shown in the previous section that in both ring loading cases, the vertical displacement w at $\xi=0$ involves evaluation of the integral

$$\int_0^\infty \beta(\xi)\,d\xi$$

In turn this involves the evaluation of the two integrals I_1 and I_2 where

$$I_1 = \int_0^\infty \frac{e^{-G}\cos G}{\sqrt[4]{r}}\,d\xi$$
$$I_2 = \int_0^\infty \frac{e^{-G}\sin G}{\sqrt[4]{r}}\,d\xi \qquad \text{(A-80)}$$

where $$G = \frac{1}{p_1}\sqrt{\frac{2m}{h}}\sqrt[4]{1+p_1^2}\left(\sqrt{r}-\sqrt{a}\right)$$

Although these integrals cannot be evaluated in closed form, it will now be shown that a very accurate approximation can be found for I and I_2 by considering the asymptotic value of an equivalent integral.

Let $$I = I_1 + iI_2 = \int_0^\infty \frac{e^{-G(1-i)}}{\sqrt[4]{r}}\,d\xi \qquad \text{(A-82)}$$

Let $$G = g\left(\sqrt{r}-\sqrt{a}\right)$$

where $$g = \frac{1}{p_1}\sqrt{\frac{2m}{h}}\sqrt[4]{1+p_1^2}$$

Then $$dG = \frac{a p_1 g}{2\sqrt{r}}\,d\xi \ \text{, and } \sqrt[4]{r} = \frac{1}{\sqrt{g}}\sqrt{G+\gamma}$$

where $$\gamma = g\sqrt{a} \qquad \text{(A-83)}$$

Substitution of these relations in (A-82) and letting $G+\gamma = t$, yields the equivalent form

$$I = \frac{2e^{\gamma(1-i)}}{a p_1 g^{3/2}}\int_\gamma^\infty \sqrt{t}\,e^{-t(1-i)}\,dt \qquad \text{(A-84)}$$

Since γ is very large, we can find an asymptotic approximation for (A-84). Successive integration by parts, yields

$$\int_\gamma^\infty \sqrt{t}\,e^{-t(1-i)}\,dt \approx e^{-\gamma(1-i)}\sqrt{\gamma}\left\{\frac{1}{1-i} + \frac{1}{2\gamma(1-i)^2}\right\} \qquad \text{(A-85)}$$

Therefore

$$I \approx \frac{2}{\sqrt{a}\sqrt[4]{a}\,p_1 g}\left\{\frac{1}{2}(1+i) + \frac{1}{8\gamma}(1+i)^2\right\}$$

Separation of real and imaginary parts, yields the desired approximations,

$$I_1 \approx \frac{1}{2a\lambda\sqrt[4]{a(1+p_1^2)}} \qquad \text{(A-87)}$$

$$I_2 \approx \frac{1}{2a\lambda\sqrt[4]{a(1+p_1^2)}} + \frac{p_1}{8a^2\lambda^2\sqrt[4]{a}\sqrt{1+p_1^2}} \qquad \text{(A-88)}$$

VII. Superposition of the Proceding Cases to Obtain a Purely Radial Ring Load

The results for a ring load of external pressure acting in a purely radial direction can now be obtained by superposition. In fact, the case for a ring load of radial pressure P_r corresponds to adding algebraically the results for

$$P_n = P_r \sin\phi_o \qquad \mathcal{S} = -P_r \cos\phi_o \tag{A-89}$$

The basic numerical results for the case of external ring load of radial force are listed below:

1. $$M_\xi(0) = \left\{ \frac{(1-16\nu^2)p_1^2}{128a^2\lambda^3(1+p_1^2)^{3/4}} + \frac{1}{4\lambda(1+p_1^2)^{1/4}} \right\} F_r \tag{A-90}$$

2. $$Q(0^+) = \left\{ \frac{(1+4\nu)p_1}{16a\lambda\sqrt[4]{1+p_1^2}} - \frac{1}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-91}$$

$$Q(0^-) = \left\{ \frac{(1+4\nu)p_1}{16a\lambda\sqrt[4]{1+p_1^2}} + \frac{1}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-92}$$

3. $$N_\xi(0^+) = -\left\{ \frac{(1+4\nu)p_1^2}{16a\lambda\sqrt[4]{1+p_1^2}} - \frac{p_1}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-93}$$

$$N_\xi(0^-) = -\left\{ \frac{(1+4\nu)p_1^2}{16a\lambda\sqrt[4]{1+p_1^2}} + \frac{p_1}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-94}$$

4. $$N_\theta(0^+) = \left\{ \frac{(1+4\nu)p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} + \frac{p_1\nu}{2} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-95}$$

$$N_\theta(0^-) = \left\{ \frac{(1+4\nu)p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} - \frac{p_1\nu}{2} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-96}$$

5. $$u(0) = \left\{ \frac{(1+4\nu)^2p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} \frac{aF_r}{Eh\sqrt{1+p_1^2}} \tag{A-97}$$

6. $$\sigma_{\xi B} = -\left\{ \frac{3(1-16\nu^2)p_1^2}{64a^2h^2\lambda^3\sqrt[4]{1+p_1^2}} + \frac{3\sqrt[4]{1+p_1^2}}{2\lambda h^2} \right\} \frac{F_r}{\sqrt{1+p_1^2}} \tag{A-98}$$

7. $$\sigma_{\theta M}(0^+) = \left\{ \frac{(1+4\nu)p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} + \frac{p_1\nu}{2} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} \frac{F_r}{h\sqrt{1+p_1^2}} \tag{A-99}$$

$$\sigma_{\theta M}(0^-) = \left\{ \frac{(1+4\nu)p_1^2}{64a\lambda\sqrt[4]{1+p_1^2}} - \frac{p_1\nu}{2} - \frac{a\lambda\sqrt[4]{1+p_1^2}}{2} \right\} \frac{F_r}{h\sqrt{1+p_1^2}} \tag{A-100}$$

Finally,

$$W(0) - W(\infty) = \frac{u(0) - u(\infty)}{p_1} + a\left(p_1 + \frac{1}{p_1}\right)\int_0^{\infty} \beta(\xi)\, d\xi \qquad \text{(A-101)}$$

However, in this case,

$$\frac{u(\infty)}{p_1} = 0 , \qquad \text{(A-102)}$$

and

$$\int_0^{\infty} \beta(\xi)\, d\xi = A_{0r} I_1 + A_{1r} I_2 \qquad \text{(A-103)}$$

where

$$A_{0r} = -\frac{(1+4\nu)\, p_1 a \sqrt[4]{a(1+p_1^2)}\,\lambda F_r}{8Eh\sqrt{1+p_1^2}} \qquad \text{(A-104)}$$

$$A_{1r} = \left\{ -\frac{(1+4\nu)\, p_1 a \sqrt[4]{a(1+p_1^2)}\,\lambda}{8Eh\sqrt{1+p_1^2}} + \frac{a^2 \sqrt[4]{a}\, \lambda^2}{Eh} \right\} F_r \qquad \text{(A-105)}$$

Therefore,

$$W(0) - W(\infty) = \left\{ -a\nu\sqrt{1+p_1^2} + \frac{a^2 p_1 \lambda}{\sqrt[4]{1+p_1^2}} + \frac{(1+4\nu)(4\nu - p_1^2)\, p_1}{32\lambda (1+p_1^2)^{3/4}} \right\} \frac{F_r}{2Eh} \qquad \text{(A-106)}$$

VIII. Cone under external pressure, P_E.

For a cone under external pressure P_E, the well-known membrane solution applies. In particular,

$$\beta_m = 0$$
$$\psi_m = -\frac{m r^2 p_1 P_E}{2Eh^2} \qquad \text{(A-107)}$$

Thus,

$$N_{\xi m} = -\frac{r P_E}{2}\sqrt{1+p_1^2} \qquad \text{(A-108)}$$

$$N_{\theta m} = -r P_E \sqrt{1+p_1^2} \qquad \text{(A-109)}$$

The displacements can be caluclated as follows:

$$u_{\xi=0} = a(\epsilon_{\theta m})_{\xi=0} = \frac{a}{Eh}\left(N_{\theta m} - \nu N_{\xi m}\right)_{\xi=0}$$
$$= -\frac{a^2 P_E \sqrt{1+p_1^2}}{Eh}\left(1 - \frac{\nu}{2}\right) \qquad \text{(A-110)}$$

Also, we have

$$W' = a\,\epsilon_{\xi m} \qquad \text{(A-111)}$$

Thus,

$$W(0) - W(R) = -a\int_0^R e_{\xi m}\, d\xi = \frac{a^2(1/2 - \nu)\sqrt{1+p_1^2}}{Eh}\left(\frac{p_1 R^2}{2} + R\right) P_E . \qquad \text{(A-112)}$$

REFERENCES

1 E. Reissner, "On the Theory of Thin Elastic Shells"; Reissner Anniversary Volume, J. W. Edwards, Ann Arbor, Michigan (1949).

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